Regularization of Gamma_1-structures in dimension 3

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FRANÇOIS LAUDENBACH AND GAËL MEIGNIEZ

Abstract. For Γ_1-structures on 3-manifolds, we give a very simple proof of Thurston's regularization theorem, first proved in [13], without using Mather's homology equivalence. Moreover, in the co-orientable case, the resulting foliation can be chosen of a precise kind, namely an "open book foliation modified by suspension". There is also a model in the non co-orientable case.

1. Introduction

A Γ_1-structure ξ, in the sense of A. Haefliger, on a manifold M is given by a line bundle $\nu = (E \to M)$, called the *normal bundle* to ξ, and a germ of codimension-one foliation $\mathcal{F}$ along the zero section, which is required to be transverse to the fibers (see [8]). To fix ideas, consider the co-orientable case, that is, the normal bundle is trivial: $E \cong M \times \mathbb{R}$; for the general case see section 7. The Γ_1-structure ξ is said to be *regular* when the foliation $\mathcal{F}$ is transverse to the zero section, in which case the pullback of $\mathcal{F}$ to M is a genuine foliation on M. A homotopy of ξ is defined as a Γ_1-structure on $M \times [0,1]$ inducing ξ on $M \times \{0\}$. A *regularization theorem* should claim that any Γ_1-structure is homotopic to a regular one. It is not true in general. An obvious necessary condition is that ν must embed into the tangent bundle τM. When ν is trivial and $\dim M = 3$ this condition is fulfilled.

The C^∞ category is understood in the sequel, unless otherwise specified. In particular M is C^∞. One calls ξ a Γ_1^r-structure ($r \geq 1$) if it is tangentially C^∞ and transversely C^r, that is, the foliation charts are C^r in the direction transverse to the leaves. We will prove the following theorem.

Theorem 1.1. *If M is a closed 3-manifold and ξ a Γ_1^r-structure, $r \geq 1$, whose normal bundle is trivial, then ξ is homotopic to a regular Γ_1^r-structure.*

Moreover, the resulting foliation of M may have its tangent plane field in a prescribed homotopy class (see proposition 6.1).

This theorem is a particular case of a general regularization theorem due to W. Thurston (see [13]). Thurston's proof was based on the deep result due to J. Mather [9], [10]: the homology equivalence between the classifying space of the group $\mathrm{Diff}_c(\mathbb{R})$ endowed with the discrete topology and the loop space $\Omega \mathrm{B}(\Gamma_1)_+$. We present a proof of this regularization theorem which does not need this result. A regularization theorem in all dimensions, still avoiding any difficult result, is provided in [12]. But there are reasons for considering the dimension 3 separately.

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Our proof provides models realizing each homotopy class of Γ_1-structure. The models are based on the notion of *open book decomposition.* Recall that such a structure on M consists of a link B in M, called the *binding*, and a fibration $p : M \smallsetminus B \to S^1$ such that, for every $\theta \in S^1$, $p^{-1}(\theta)$ is the interior of an embedded surface, called the *page* P_θ, whose boundary is the binding. The existence of open book decomposition could be proved by J. Alexander when M is orientable, as a consequence of [1] (every orientable closed 3-manifold is a branched cover of the 3-sphere) and [2] (every link can be braided); but he was ignoring this concept which was introduced by H. Winkelnkemper in 1973 [16]. Henceforth, we refer to the more flexible construction by E. Giroux, which includes the non-orientable case (see section 3). An open book gives rise to a foliation $\mathcal{O}$ constructed as follows. The pages endow B with a normal framing. So a tubular neighborhood T of B is trivialized: $T \cong B \times D^2$. Out of T the leaves are the pages modified by spiraling around T; the boundary of T is a union of compact leaves; and the interior of T is foliated by a Reeb component, or a generalized Reeb component in the sense of Wood [17]. For technical reasons in the homotopy argument of section 4, the Reeb components of $\mathcal{O}$, instead of being usual Reeb components, will be *thick Reeb components* in which a neighborhood of the boundary is foliated by toric compact leaves. We call such a foliation an *open book foliation.*

The latter can be modified by inserting a so called *suspension foliation.* Precisely, let Σ be a compact sub-surface of some leaf of $\mathcal{O}$ out of T and $\Sigma \times [-1, +1]$ be a foliated neighborhood of it (each $\Sigma \times \{t\}$ being contained in a leaf of $\mathcal{O}$). Let $\varphi : \pi_1(\Sigma) \to \mathrm{Diff_c}(]-1, +1[)$ be some representation into the group of compactly supported diffeomorphisms; φ is assumed to be trivial on the peripheral elements. It allows us to construct a *suspension foliation* $\mathcal{F}_\varphi$ on $\Sigma \times [-1, +1]$, whose leaves are transverse to the vertical segments $\{x\} \times [-1, +1]$ and whose holonomy is φ. The modification consists of removing $\mathcal{O}$ from the interior of $\Sigma \times [-1, +1]$ and replacing it by $\mathcal{F}_\varphi$. The new foliation, denoted $\mathcal{O}_\varphi$, is an *open book foliation modified by suspension.* Theorem 1.1 can now be made more precise:

Theorem 1.2. *Every co-orientable Γ_1^r-structure, $r \geq 1$, is homotopic to an open book foliation modified by suspension.*

The proof of this theorem is given in sections 2 - 4 when $r \geq 2$. In section 5, we explain how to get the less regular case $1 \leq r < 2$. We have chosen to treat the case $r = 1 + bv$ (the holonomy local diffeomorphisms are C^1 and their first derivatives have a bounded variation). Indeed, Mather observed in [11] that $Diff_c^{1+bv}(\mathbb{R})$ is not a perfect group and it is often believed that the perfectness of $Diff_c^r(\mathbb{R})$ plays a role in the regularization theorem.

In section 6, the homotopy class of the tangent plane field will be discussed. Finally the case of non co-orientable Γ_1^r-structure will be sketched in section 7 where the corresponding models, based on *twisted open book*, will be presented.

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2. Tsuboi's construction

A Γ_1-structure ξ on M is said to be *trivial* on a codimension 0 submanifold W when, for every $|t|$ small enough, $W \times \{t\}$ lies in a leaf of the associated foliation.

Every closed 3-manifold M has a so-called *Heegaard decomposition* $M = H_- \underset{\Sigma}{\cup} H_+$, where $H_\pm$ is a possibly non-orientable handlebody (a ball with handles of index 1 attached) and Σ is their common boundary. A *thick Heegaard decomposition* is a similar decomposition where the surface is thickened:

$$M = H'_- \underset{\Sigma\times\{-1\}}{\cup} \Sigma \times [-1, +1] \underset{\Sigma\times\{+1\}}{\cup} H'_+.$$

The following statement is due to T. Tsuboi in [14] where it is left to the reader as an exercise.

Proposition 2.1. *Given a Γ_1-structure ξ of class C^r, $r \geq 2$, on a closed 3-manifold M, there exists a thick Heegaard decomposition and a homotopy $(\xi_t)_{t\in[0,1]}$ from ξ such that:*
1) ξ_1 is trivial on $H'_\pm$;
2) ξ_1 is regular on $\Sigma \times [-1, +1]$ and the induced foliation is a suspension.

Proof. With ξ and its foliation $\mathcal{F}$ defined on an open neighborhood of the zero section $M \times 0$ in $M \times \mathbb{R}$, there comes a covering of the zero section by boxes, open in $M \times \mathbb{R}$, bi-foliated with respect to $\mathcal{F}$ and the fibers. We choose a C^1-triangulation Tr of M so fine that each simplex lies entirely in a box. With Tr comes a vector field X defined as follows.

First, on the standard k-simplex there is a smooth vector field X_{Δ^k}, tangent to each face, which is the (descending) gradient of a Morse function having one critical point of index k at the barycenter and one critical point of index i at the barycenter of each i-face. When $\Delta^i \subset \Delta^k$ is an i-face, X_{Δ^i} is the restriction of X_{Δ^k} to Δ^i. Now, if σ is a k-simplex of Tr, thought of as a C^1-embedding $\sigma : \Delta^k \to M$, we define $X_\sigma := \sigma_*(X_{\Delta^k})$. The union of the X_σ's is a C^0 vector field X which is uniquely integrable. After a reparametrization of each simplex we may assume that the stable manifold $W^s(b(\sigma))$ of the barycenter $b(\sigma)$ is C^1.

The Γ_1-structure ξ (co-oriented by the $\mathbb{R}$ factor of $M \times \mathbb{R}$) is said to be in *Morse position* with respect to Tr if:

(i) it has a smooth Morse type singularity of index k at the barycenter of each k-simplex and it is regular elsewhere;
(ii) X is (negatively) transverse to ξ out of the singularities.

Lemma 2.2. *Let $\mathcal{F}$ be the foliation associated to ξ. There exists a smooth section s such that $s^*\mathcal{F}$ is in Morse position with respect to Tr.*

Note that, as s is homotopic to the zero section, the Γ_1-structure $s^*\mathcal{F}$ on M is homotopic to ξ.

Proof. Assume that s is already built near the $(k-1)$-skeleton. Let σ be a k-simplex. We explain how to extend s on a neighborhood of σ. After a fibered isotopy of $M \times \mathbb{R}$ over the identity of M, we may assume that $\mathcal{F}$ is trivial near $\{b(\sigma)\} \times \mathbb{R}$. Now, near $b(\sigma)$, we ask s to coincide with the graph of some local positive Morse function f_σ whose Hessian is negative definite on $T_{b(\sigma)}\sigma$ and positive definite on $T_{b(\sigma)}W^s(b(\sigma))$. This function is now fixed up to a positive constant factor. We will extend s as the graph of some function h in the $\mathcal{F}$-foliated chart over a neighborhood of σ. This function is already given on a neigborhood $N(\partial\sigma)$ of $\partial\sigma$ where it is C^r, the regularity of ξ, and satisfies $X.h < 0$ except at the barycenter of each face.

On the one hand, choose an arbitrary extension h_0 of h to a neighborhood of σ vanishing near $b(\sigma)$. On the other hand, choose a nonnegative function g_σ such that:

- $g_\sigma = 0$ near $\partial\sigma$;
- $g_\sigma = f_\sigma$ near $b(\sigma)$;
- $X.g_\sigma < 0$ when $g_\sigma > 0$ except at $b(\sigma)$.

Then, if $c > 0$ is a large enough constant, $h := h_0 + cg_\sigma$ has the required properties, except smoothness. Returning to $M \times \mathbb{R}$, the section s we have built is C^r, smooth near the singularities, and X is transverse to $s^*\xi$ except at the singularities. Therefore, there exists a smooth C^r-approximation of s, relative to a neighborhood of the barycenters which meets all the required properties. □

Thus, by a deformation of the zero section which induces a homotopy of ξ, we have put ξ in *Morse position* with respect to Tr. In the same way, applying lemma 2.2 to the trivial Γ_1-structure ξ_0, we also have a Morse function f such that $X.f < 0$ except at the barycenters.

Let G_- (resp. G_+) denote the closure of the union of the unstable (resp. stable) manifolds of the singularities of X of index 1 (resp. 2). The following properties are clear:

(a) In M, the subset G_- (resp. G_+) is a C^1-complex of dimension 1.
(b) It admits arbitrarily small handlebody neighborhoods H'_- (resp. H'_+) whose boundary is transverse to X.
(c) Every orbit of X outside $H'_\pm$ has one end point on $\partial H'_-$ and the other on $\partial H'_+$. This also holds true for any smooth C^0-approximation $\widetilde{X}$ of X (in particular $\widetilde{X}$ is still negatively transverse to ξ).

Given a (co-orientable) Γ_1-structure ξ on a space G, by an *upper (resp. lower) completion* of ξ one means a foliation $\mathcal{F}$ of $G \times (-\epsilon, 1]$ (resp. $G \times [-1, \epsilon)$), for some positive ϵ, which is transverse to every fiber $\{x\} \times (-\epsilon, 1]$ (resp. $\{x\} \times [-1, \epsilon)$), whose germ along $G \times \{0\}$ is ξ, and such that $G \times \{t\}$ is a leaf of $\mathcal{F}$ for every t close enough to $+1$ (resp. -1).

Lemma 2.3. *Every co-orientable Γ_1^r-structure on a simplicial complex G of dimension 1, $r \geq 2$, admits an upper (resp. lower) completion of class C^r.*

Proof. One reduces immediately to the case where G is a single edge. In that case, using a partition of unity, one builds a line field which fulfills the claim. This line field is integrable. □

By (a), the Γ_1-structure ξ admits an upper (resp. lower) completion over G_+ (resp. G_-), and thus also over an open neighborhood N_+ (resp. N_-) of G_+ (resp. G_-). By (b), there is a handlebody neighborhood $H'_\pm$ of $G_\pm$ contained in $N_\pm$ and whose boundary is transverse to X. So we have a foliation $\mathcal{F}$ defined on a neighborhood of

$$(M \times \{0\}) \cup (H'_- \times [-1, 0]) \cup (H'_+ \times [0, 1])$$

which is transverse to X on $M \smallsetminus (H'_- \cup H'_+)$ and tangent to $H'_\pm \times \{t\}$ for every t close to ± 1.

By (c), there is a diffeomorphism $F : M \smallsetminus Int(H'_- \cup H'_+) \to \Sigma \times [-1, +1]$ for some closed surface Σ, which maps orbit segments of $\widetilde{X}$ onto fibers.

For a small $\epsilon > 0$, choose a function $\psi : \mathbb{R} \to [-1, +1]$ which is smooth, odd, and such that:

- $\psi(t) = 0$ for $0 \leq t \leq 1 - 3\epsilon$ and $\psi(1 - 2\epsilon) = \epsilon$;
- ψ is affine on the interval $[1 - 2\epsilon, 1 - \epsilon]$;

- $\psi(1-\epsilon) = 1-\epsilon$ and $\psi(t) = 1$ for $t \geq 1$;
- $\psi' > 0$ on the interval $]1-3\epsilon, 1[$.

Let $s : M \to M \times \mathbb{R}$ be the graph of the function whose value is ± 1 on $H'_\pm$ and $\psi(t)$ at the point $F^{-1}(x,t)$ for $(x,t) \in \Sigma \times [-1,+1]$. When ϵ is small enough, it is easily checked that, for every $x \in \Sigma$, the path $t \mapsto s \circ F^{-1}(x,t)$ is transverse to $\mathcal{F}$ except at its end points. Then, $\xi_1 := s^*\mathcal{F}$ is homotopic to ξ and obviously fulfills the conditions required in proposition 2.1. $\square$

3. Giroux's construction

We use here theorem III.2.7 from Giroux's article [5], which states the following:

> *Let M be a closed 3-manifold (orientable or not). There exist a Morse function f : $M \to \mathbb{R}$ and a co-orientable surface S which is f-essential in M.*

Giroux says that S is *f-essential* when the restriction $f|S$ has exactly the same critical points as f and the same local *extrema*. In the sequel, we call such a surface a *Giroux surface*.

Giroux explained to us [6] how this notion is related to open book decompositions. In the above statement, the function f can be easily chosen *self-indexing* (the value of a critical point is its Morse index in M). Thus, let N be the level set $f^{-1}(3/2)$. The smooth curve $B := N \cap S$ will be the binding of the open book decomposition we are looking for. It can be proved that the following holds for every regular value a, $0 < a \leq 3/2$:

- the level set $f^{-1}(a)$ is the union along their common boundaries of two surfaces, N_1^a and N_2^a, each one being diffeomorphic to the sub-level surface $S^a := S \cap f^{-1}([0,a])$;
- the sub-level $M^a := f^{-1}([0,a])$ is divided by S^a into two parts P_1^a and P_2^a which are isomorphic handlebodies (with corners);
- S^a is isotopic to N_i^a through P_i^a, for $i = 1, 2$, by an isotopy fixing its boundary curve $S^a \cap f^{-1}(a)$.

This claim is obvious when a is small and the property is preserved when crossing the critical level 1. In this way the handlebody $H_- := f^{-1}([0,3/2])$ is divided by $S^{3/2}$ into two diffeomorphic parts $P_i^{3/2}$, $i = 1, 2$, and we have $N = N_1^{3/2} \cup N_2^{3/2}$. We take $S^{3/2}$, which is isotopic to $N_i^{3/2}$ in $P_i^{3/2}$, as a page. The figure is the same in $H_+ := f^{-1}([3/2,3])$. The open book decomposition is now clear.

Proposition 3.1. *Let $K \subset M$ be a compact connected co-orientable surface whose boundary is not empty. Then there exists an open book decomposition whose some page contains K in its interior.*

Proof. (Giroux) According to the above discussion it is sufficient to find a Morse function f and a Giroux surface S (with respect to f) containing K. Let H_0 be the quotient of $K \times [-1,1]$ by shrinking to a point each interval $\{x\} \times [-1,1]$ when $x \in \partial K \times [-1,1]$. After smoothing, it is a handlebody whose boundary is the double of K. On H_0 there exists a standard Morse function f_0 which is constant on ∂H_0, having one minimum, the other critical points being of index 1. The surface $K \times \{0\}$ can be made f_0-essential. This function is then extended to a global Morse function $\tilde{f}_0$ on M. At this point we have to follow the proof of Theorem III.2.7 in [5]. The function $\tilde{f}_0$ is changed on the complement of H_0, step by step when crossing its critical level, so that $K \times \{0\}$ extends as a Giroux surface in M. □

Let now ξ be a Γ_1-structure meeting the conclusion of proposition 2.1, up to rescaling the interval to $[-\varepsilon, +\varepsilon]$. Let $\mathcal{F}_\varphi$ be the suspension foliation induced by ξ on $\Sigma \times [-\varepsilon, +\varepsilon]$. Choose $x_0 \in \Sigma \times \{0\}$; the segment $x_0 \times [-\varepsilon, +\varepsilon]$ is transverse to $\mathcal{F}_\varphi$. Let K be the surface obtained from $\Sigma \times 0$ by removing a small open disk centered at x_0. The foliation $\mathcal{F}_\varphi$ foliates $K \times [-\varepsilon, +\varepsilon]$ so that $K \times \{t\}$ lies in a leaf, when t is close to $\pm\varepsilon$, and $\partial K \times [-\varepsilon, +\varepsilon]$ is foliated by parallel circles. We apply proposition 3.1 to this K.

Corollary 3.2. *There exists an open book foliation $\mathcal{O}$ of M inducing the trivial foliation on $K \times [-\varepsilon, +\varepsilon]$ (the leaves are $K \times \{t\}$, $t \in [-\varepsilon, +\varepsilon]$).*

Therefore, we have an open book foliation modified by suspension by replacing the above trivial foliation of $K \times [-\varepsilon, +\varepsilon]$ by $\widetilde{\mathcal{F}}_\varphi$, the trace of $\mathcal{F}_\varphi$ on $K \times [-\varepsilon, +\varepsilon]$. Let $\mathcal{O}_\varphi$ be the resulting foliation of M and ξ_φ be its regular Γ_1-structure. For proving theorem 1.2 (when $r \geq 2$) it is sufficient to prove that ξ and ξ_φ are homotopic. This is done in the next section.

4. Homotopy of Γ_1-structures

We are going to describe a homotopy from ξ_φ to ξ. Recall the tube T around the binding. For simplicity, we assume that each component of T is foliated by a standard Reeb foliation; the same holds true if T is foliated by Wood components (in the sense of [17]). Let T' be a slightly larger tube.

Lemma 4.1. *There exists a homotopy, relative to $M \smallsetminus int(T')$, from ξ_φ to a new Γ_1-structure ξ_1 on M such that:*
1) ξ_1 is trivial on T;
2) ξ_1 is regular on $T' \smallsetminus int(T)$ with compact toric leaves near ∂T and spiraling half-cylinder leaves with boundary in $\partial T'$ (as in an open book foliation).

Proof. Recall from the introduction that we only use *thick* Reeb components. So there is a third concentric tube T'', $T \subset int(T'') \subset int(T')$, so that $T'' \smallsetminus int(T)$ is foliated by toric leaves; and $int(T)$ is foliated by planes.

On each component of ∂T we have coodinates (x, y) coming from the framing of the binding, the x-axis being a parallel and the y-axis being a meridian. Let γ be a parallel in ∂T. The Γ_1-structure which is induced by ξ_φ on γ is singular but not trivial and the germ of foliation $\mathcal{G}$ along the zero section in the normal line bundle $A \cong \gamma \times \mathbb{R}$ is shown on figure 1.

The annulus A is endowed with coordinates $(x, z) \in \gamma \times \mathbb{R}$. The orientation of the z-axis, which is also the orientation of the normal bundle to the foliation $\mathcal{O}_\varphi$ along γ, points to the interior of T. So the leaves of $\mathcal{G}$ are parallel circles in $\{z \leq 0\}$ and spiraling leaves in $\{z > 0\}$. Take coordinates (x, y, r) on T where r is the distance to the binding; say that $r = 1$ on $\partial T''$ and $r = 1/2$ on ∂T. Let $\lambda(r^2)$ be an even smooth function with $r = 0$ as unique critical point, vanishing at $r = 1/2$ and $\lambda(1) < 0 < \lambda(0)$. Consider $g : T'' \to A$, $g(x, y, r) = \big(x, \lambda(r^2)\big)$.

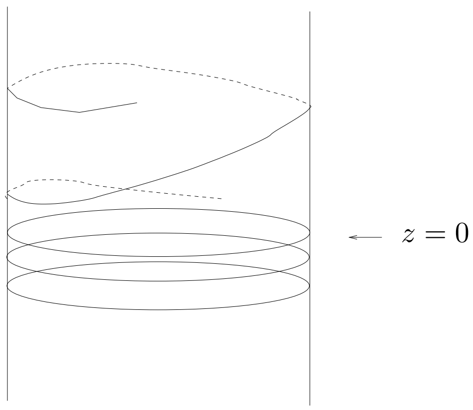

Figure 1

It is easily seen that $\xi_\varphi | int(T'') \cong g^*\mathcal{G}$. Let now $\bar\lambda(r^2)$ be a new even function coinciding with $\lambda(r^2)$ near $r = 1$, having negative values everywhere and whose critical set is $\{r \in [0, 1/2]\}$. Let $\bar g : T'' \to A$, $\bar g(x, y, r) = \big(x, \bar\lambda(r^2)\big)$. A barycentric combination of λ and $\bar\lambda$ yields a homotopy from g to $\bar g$ which is relative to a neighborhood of $\partial T''$. The Γ_1-structure ξ_1 we are looking for is defined by $\xi_1 | int(T'') = \bar g^*(\mathcal{G})$ and $\xi_1 = \xi_\varphi$ on a neighborhood of $M \smallsetminus int(T'')$. $\square$

Recall the domain $K \times [-\varepsilon, +\varepsilon]$ from the previous section. After the following lemma we are done with the homotopy problem.

Lemma 4.2. *There exits a homotopy from ξ_1 to ξ relative to $K \times [-\varepsilon, +\varepsilon]$.*

Proof. Let us denote $M' := M \smallsetminus int(K \times [-\varepsilon, +\varepsilon])$ which is a manifold with boundary and corners. It is equivalent to prove that the restrictions of ξ_1 and ξ to M' are homotopic relatively to $\partial M'$. Consider the standard closed 2-disk $D = D^2$ endowed with the Γ_1-structure ξ_D which is shown on figure 2.

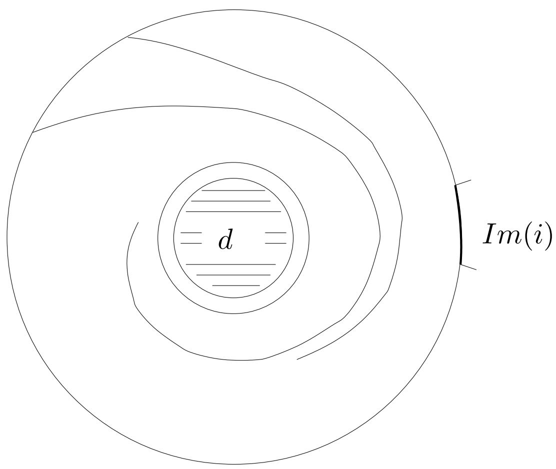

Figure 2

It is trivial on the small disk d and regular on the annulus $D \smallsetminus int(d)$. In the regular part, the leaves are circles near ∂d and the other leaves are spiraling, crossing ∂D transversely. One checks that the restrition of ξ_1 to M' has the form $f^*\xi_D$ from some map $f : M' \to D$. We take $f|T : T \to d$ to be the open book trivialization of T (recall the binding has a canonical framing); $f|\partial K \times [-\varepsilon, +\varepsilon]$ to be the projection pr_2 onto $[-\varepsilon, +\varepsilon]$ composed with an embedding $i : [-\varepsilon, +\varepsilon] \to \partial D$ and f maps each leaf of the regular part of ξ_1 to a leaf of the regular part of ξ_D. As K does not approach T, we can take $f(\partial M') = i([-\varepsilon, +\varepsilon])$; actually, except near T, f is given by the fibration over $S^1 = \partial D^2$ of the open book decomposition.

Once ξ has Tsuboi's form (according to proposition 2.1), the restrition of ξ to M' has a similar form: $\xi = k^*\xi_D$ for some map $k : M' \to D^2$. Recall that M' is the union of two handlebodies and a solid cylinder $D^2 \times [-\varepsilon, +\varepsilon]$. Take k to be $i \circ pr_2$ on the cylinder and k to be constant on each handlebody. Observe that f and k coincide on $\partial M'$. As D retracts by deformation onto the image of i, one deduces that f and k are homotopic relatively to $\partial M'$. □

This finishes the proof of theorem 1.2 when $r \geq 2$.

5. The case C^{1+bv}

A co-oriented Γ_1^r-structure ξ on M can be realized by a foliation $\mathcal{F}$ defined on a neighborhood of the 0-section in $M \times \mathbb{R}$; it is made of bi-foliated charts which are C^∞ in the direction of the leaves and C^r in the direction of the fibers. Consider such a box $\mathcal{U}$ over an open disk D centered at $x_0 \in M$; its trace on the x_0-fiber is an interval I. Each leaf of $\mathcal{U}$ reads $z = f(x,t)$, $x \in D$, for some $t \in I$. Here f is a function which is smooth in x and C^r in t, with $f(x_0, t) = t$; the foliating property is equivalent to $\frac{\partial f}{\partial t} > 0$. When $r = 1 + bv$, there is a positive measure $\mu(x,t)$ on I, without atoms and depending smoothly on x, such that:

$$(*) \qquad \frac{\partial}{\partial t} f(x,t) - \frac{\partial}{\partial t} f(x,t_0) = \int_{t_0}^{t} \mu(x,t).$$

Proposition 5.1. *Theorem 1.2 holds true for any class of regularity $r \geq 1$ including the class $r = 1 + bv$.*

Proof. The only part of the proof which requires some care of regularity is section 2, especially the proof of lemma 2.3. Indeed, we have to avoid integrating C^0 vector fields. For proving lemma 2.3 with weak regularity we use the lemmas below which we shall prove in the case $r = 1 + bv$ only.

Lemma 5.2. *Let $f : D \times I \to \mathbb{R}$ be a C^r-function as above. Assume $I =]-2\varepsilon, +\varepsilon[$ and $f(x, -2\varepsilon) > -1$ for every $x \in D$. Then there exists a function $F : D \times [-1, 0] \to \mathbb{R}$ of class C^r such that:*

1) *$F(x,t) = f(x,t)$ when $t \in [-\varepsilon, 0]$,*
2) *$F(x,t) = t$ when t is close to -1,*
3) *$\dfrac{\partial F}{\partial t} > 0$.*

Proof. Let $\mu(x,t)$ be the positive measure whose support is $[-\varepsilon,0]$ such that formula $(*)$ holds for every $(x,t)\in D\times[-\varepsilon,0]$ and $t_0=0$. There exists another positive measure $\nu(x,t)$, smooth in x and whose support is contained in $]-1,-\varepsilon]$, such that

$$(**)\qquad f(x,-\varepsilon)=-1+\int_{-1}^{-\varepsilon}\left(\int_{-1}^{t}\nu(x,\tau)\right)dt.$$

Then a solution is

$$F(x,t)=-1+\int_{-1}^{t}\left(\int_{-1}^{s}\left(\mu(x,\tau)+\nu(x,\tau)\right)\right)ds.$$

□

Lemma 5.3. *Let A_1 and A_2 two disjoint closed sub-disks of D. Let F_1 and F_2 be two solutions of lemma 5.2. Then there exists a third solution which equals F_1 when $x\in A_1$ and F_2 when $x\in A_2$.*

Proof. Both solutions F_1 and F_2 differ by the choice of the measure $\nu(x,t)$ in formula $(**)$, which is ν_i for F_i. Choose a partition of unity $1=\lambda_1(x)+\lambda_2(x)$ with $\lambda_i=1$ on A_i. Then $\nu(x,t)=\lambda_1(x)\nu_1(x,t)+\lambda_2(x)\nu_2(x,t)$ yields the desired solution. □

The proof of proposition 5.1 is now easy. As already said, it is sufficient to prove lemma 2.3 in class C^r, $r\geq 1$. It is an extension problem of a foliation given near the 1-skeleton $Tr^{[1]}\times\{0\}$ to $Tr^{[1]}\times[-1,0]$. One covers $Tr^{[1]}$ by finitely many n-disks D_j. The problem is solved in each $D_j\times[-1,0]$ by applying lemma 5.2. By applying lemma 5.3 one makes the different extensions match together. □

6. Homotopy class of plane fields

It is possible to enhance theorem 1.1 by prescribing the homotopy class of the underlying co-oriented plane field (see proposition 6.1 below). The question of doing the same with respect to theorem 1.2 is more subtil (see proposition 6.3).

Proposition 6.1. *Given a co-oriented Γ_1-structure ξ on the closed 3-manifold M and a homotopy class $[\nu]$ of co-oriented plane field in the tangent space τM, there exists a regular Γ_1-structure ξ_{reg} homotopic to ξ whose underlying foliation $\mathcal{F}_{reg}\cap M$ has a tangent co-oriented plane field in the class $[\nu]$.*

Before proving it we first recall some well-known facts on co-oriented plane fields (see [4]). Given a base plane field ν_0, a suitable Thom-Pontryagin construction yields a natural bijection between the set of homotopy classes of plane fields on M and $\Omega_1^{\nu_0}(M)$, the group of (co)bordism classes of ν_0-framed and oriented closed (maybe non-connected) curves in M. A ν_0-framing of the curve γ is an isomorphism of fiber bundles $\varepsilon:\nu(\gamma,M)\to\nu_0|\gamma$, whose source is the normal bundle to γ in M. We denote γ^ε the curve endowed with this framing. Moreover, given γ^ε, if γ' is homologous to γ in M there exists a ν_0-framing ε' such that $(\gamma')^{\varepsilon'}$ is cobordant to γ^ε.

Proof of 6.1. We can start with an open book foliation $\mathcal{O}_\varphi$ yielded by theorem 1.2. Let ν_0 be its tangent plane field. Near the binding, the meridian loops (out of T) are transverse to $\mathcal{O}_\varphi$ and homotopic to zero in M. As a consequence, each 1-homology class may be represented as well by a (multi)-curve in a page or by a connected curve out of T positively transverse to all pages. We do the second choice for γ^ε, the ν_0-framed curve whose cobordism class encodes $[\nu]$ with respect to ν_0.

Hence we are allowed to *turbulize* $\mathcal{O}_\varphi$ along γ. In a small tube $T(\gamma)$ about γ, we put a Wood component. Outside, the leaves are spiraling around $\partial T(\gamma)$. Let $\mathcal{O}_\varphi^{turb}$ be the resulting foliation. Whatever the chosen type of Wood component is, the Γ_1-structures of $\mathcal{O}_\varphi^{turb}$ and $\mathcal{O}_\varphi$ are homotopic by arguing as in section 4. But the framing ε tells us which sort of Wood component will be convenient for getting the desired class $[\nu]$ (see lemma 6.1 in [17]). ☐

In the previous statement, we have lost the nice model we found in theorem 1.2. Actually, thanks to a lemma of Vincent Colin [3], it is possible to recover our model, at least when M is orientable (see below proposition 6.3).

Lemma 6.2. (Colin) *Let (B, p) be an open book decomposition of M and γ be a simple connected curve in some page P. Assume γ is orientation preserving. Then there exist a positive stabilization (B', p') of (B, p) and a curve γ' in B' which is isotopic to γ in M. When γ is a multi-curve, the same holds true after a sequence of stabilizations.*

The *positive Hopf open book decomposition* of the 3-sphere is the one whose binding is made of two unknots with linking number $+1$; a page is an annulus foliated by fibers of the Hopf fibration $S^3 \to S^2$. A positive stabilization is a "connected sum" with this open book. The new page P' is obtained from P by *plumbing* an annulus A whose core bounds a disk in M (see [7] for more details and other references).

Proof. If γ is connected, only one stabilization is needed. We are going to explain this case only. A tubular neighborhood of γ in P is an annulus.

Choose a simple arc α in P joining γ to some component β of B without crossing γ again. Let $\tilde\gamma$ be a simple arc from β to itself which follows $\alpha^{-1} * \gamma * \alpha$. The orientation assumption implies that the surgery of β by $\tilde\gamma$ in P provides a curve with two connected components, one of them being isotopic to γ in P. Let P_π be the page opposite to P and $R : P \to P_\pi$ the time π of a flow transverse to the pages (and stationary on B). The core curve C of the annulus A that we use for the plumbing is the union $\tilde\gamma \cup R(\tilde\gamma)$. And A is $(+1)$-twisted around C (with respect to its unknot framing) as in the Hopf open book. Let H be the 1-handle which is the closure of $A \setminus P$. Surgering B by H provides the new binding. By construction, one of its components is isotopic to γ. ☐

Proposition 6.3. *Let $\mathcal{O}_\varphi$ be an open book foliation modified by suspension, whose its underlying open book is denoted (B, p). Let ν_0 be its tangent co-oriented plane field. Let γ^ε be a ν_0-framed curve in M and $[\nu]$ be its associated class of plane field. Assume γ is orientation preserving. Then there exists an open book foliation $\mathcal{O}'_\varphi$ with the following properties:*

1) its tangent plane field is in the class $[\nu]$;

2) the suspension modification is the same for $\mathcal{O}'_\varphi$ as for $\mathcal{O}_\varphi$ and is supported in $K \times [-\varepsilon, +\varepsilon]$;

3) as Γ_1-structures, $\mathcal{O}_\varphi$ and $\mathcal{O}'_\varphi$ are homotopic.

Proof. As said in the proof of 6.1, up to framed cobordism, γ^ε may be chosen as a simple (multi)-curve in one page P of (B, p). Applying Colin's lemma provides a stabilization (B', p') such that, up to isotopy, γ lies in the new binding. Observe that, if K is in P, K is still in the new page P'; hence 2) holds for any open book foliation carried by (B', p'). Once γ^ε is in the binding, for a suitable Wood component foliating a tube about γ^ε, item 1) is fulfilled. Finally item 3) follows from item 2) and the proofs in section 4. $\square$

7. Case of a Γ_1-structure with a twisted normal bundle

What happens when the bundle ν normal to ξ is twisted? It is known that a necessary condition to regularization is the existence of a fibered embedding $i : \nu \to \tau M$ into the tangent fiber bundle to M. Conversely, assuming that this condition is fulfilled, we are going to state a normal form theorem analogous to theorem 1.2. Since no step of the previous proof can be immediately adapted to this situation, we believe that it deserves a sketch of proof.

7.1. In the first step (Tsuboi's construction), we do not have "Morse position" with respect to a triangulation, since index and co-index of a singularity cannot be distinguished. Instead of lemma 2.2, we have the following statement.

> *After some homotopy, ξ has Morse singularities and admits a pseudo-gradient whose dynamics has no recurrence (that is, every orbit has a finite length).*

Here, by a pseudo-gradient, it is meant a smooth section X of $Hom(\nu, \tau M)$, a *twisted vector field* indeed, such that $X \cdot \xi < 0$ except at the singularities (this sign is well-defined whatever a local orientation of ν, or co-orientation of ξ, is chosen); such a pseudo-gradient always exists by using an auxiliary Riemannian metric.

Sketch of proof. Generically ξ has Morse singularities. Let X_0 be a first pseudo-gradient, which is required to be the usual negative gradient in Morse coordinates near each singularity. Finitely many mutually disjoint 2-disks of M are chosen in regular leaves of ξ such that every orbit of X_0 crosses one of them. Following Wilson's idea [15], ξ and X_0 are changed in a neighborhood $D^2 \times [-1, +1]$ of each disk into a *plug* such that every orbit of the modified pseudo-gradient X is trapped by one of the plugs. The plug has the mirror symmetry with respect to $D^2 \times \{0\}$. In $D^2 \times [0, 1]$ we just modify ξ by introducing a cancelling pair of singularities, center-saddle. $\square$

Let G be the closure of the one-dimensional invariant manifold of all saddles. It is a graph. We claim: *$\nu|G$ is orientable.* Indeed, we orient each edge from its saddle end point to its center end point. This is an orientation of $\nu|G$ over the complement of the vertices. It is easily checked that this orientation extends over the vertices. Thus X becomes a usual vector field near G and we have an arbitrarily small tubular neighborhood H of G whose boundary is transverse to X, and X enters H. Now, the negative completion of $\xi|H$ can be performed as in lemma 2.3.

The complement $\hat{M}$ of $int\, H$ in M is fibered over a surface Σ, the fibers being intervals ($\cong [-1, 1]$) tangent to X. By taking a section we think of Σ as a surface in $M \smallsetminus H$. Since

ξ is not co-orientable, Σ *is one-sided and* G *is connected.* Arguing as in section 2, *after some homotopy,* ξ *becomes trivial on* H *and transverse to* X *on* $\hat{M}$*, hence a suspension foliation corresponding to a representation* $\varphi : \pi_1(\Sigma) \to Diff_c(]-1,1[)$.

7.2. In the second step (Giroux's construction), we have to leave the open books and we need a *twisted open book*. It is made of the following:

- a binding B which is a 1-dimensional closed co-orientable submanifold in M;
- a *Seifert fibration* $p : M \smallsetminus B \to [-1,+1]$ which has two one-sided exceptional surface fibers $p^{-1}(\pm 1)$ and which is a proper smooth submersion over the open interval;
- when t goes to ± 1, $p^{-1}(t)$ tends to a 2-fold covering of $p^{-1}(\pm 1)$;
- near B the foliation looks like an open book.

The exceptional fibers are compactified by B as smooth surfaces with boundary. But, for $t \in]-1,+1[$, $p^{-1}(t)$ is compactified by B as a closed surface showing (in general) an angle along B. Notice that, since B is co-orientable, a twisted open book gives rise to a smooth foliation where each component of the binding is replaced by a Reeb component, the pages spiraling around it.

Such an open book is generated by a *one-sided Giroux surface*, which is the union of the compactified exceptional fibers. Abstractly, a one-sided Giroux surface in M with respect to a Morse function $f : M \to \mathbb{R}$ is a one-sided surface S such that $f|S$ has the same critical points and the same extrema as f and fulfills the extra condition: for every regular value $t \in \mathbb{R}$, $f^{-1}(t) \cap S$ is a two-sided curve in the level set $f^{-1}(t)$. Starting with (S, f) where f is a self-indexing Morse function, a twisted open book is easily constructed. Its binding is the co-orientable curve $f^{-1}(3/2) \cap S$. In general such a one-sided Giroux surface (or twisted open book) does not exist on M; the obstruction lies in the existence of a twisted line subbundle of τM. Nevertheless, with a suitable assumption, we have an analogue of proposition 3.1:

> *Let* $i : \nu \to \tau M$ *be an embedding of a twisted line bundle. Let* $K \subset M$ *be a compact connected one-sided surface whose boundary is not empty. Assume the following:* $\nu|K$ *is twisted,* $\nu|\partial K$ *is trivial and the normal bundle* $\nu(K, M)$ *is homotopic to* $i(\nu)|K$*. Then there exists a twisted open book with one exceptional page containing* K *in its interior.*

Sketch of proof. We give the proof only in the setting of 7.1 by taking K to be the closure of $\Sigma \smallsetminus d_0$ where d_0 is a small closed 2-disk in Σ. Recall the fibration $\rho : \hat{M} \to \Sigma$. Let H' be the handlebody made of H to which is glued the 1-handle $\rho^{-1}(d_0)$. Let M' be the complement of $int\, H'$. Consider a minimal system of mutually disjoint compression disks $d_1, ..., d_g$ of H, so that cutting H along them yields a ball; g is the genus of H. Thus $d_0, d_1, ..., d_g$ is a minimal system of mutually disjoint compression disks of H'.

We claim: $g + 1$ *is even.* Indeed, by assumption there exists a non vanishing section of the bundle $Hom(\nu, \tau M)$. Thus, the number of zeroes of the pseudo-gradient X is even and the Euler characteristic $\chi(G)$ of the graph G is even. As G is connected, the genus g is odd, which proves the claim.

Now, one follows Giroux's algorithm for completing K to a closed Giroux's surface. On the surface $\partial M'$ the union of the attaching curves $\partial d_0, \partial d_1, ..., \partial d_g$ is not separating. Then, after some isotopy, for each $i = 0, ..., g$, ∂d_i crosses ∂K in exactly two points a_i, b_i linked by an arc α_i

(resp. α'_i) in ∂d_i (resp. ∂K), so that $\alpha_i \cup \alpha'_i$ bounds a disk in $\partial M'$. Moreover, one can arrange that all the arcs $\alpha_0, ..., \alpha_g$ are parallel. Also we link a_i, b_i by a simple arc in d_i. Now, each compression disk defines, simultaneously, a 1-handle which is glued to K and a 2-handle which is glued to M', yielding a proper surface K_1 in some 3-submanifold M'' of M, whose complement is a ball. The boundary of K_1 is made of $g+2$ parallel curves in the sphere $\partial M''$. As this number is odd, Giroux described a process of adding cancelling pairs of 1- and 2- handles whose effect is to change K_1 into $K_2 \subset M''$ such that ∂K_2 is made of one curve only ([5], p. 676-677). Hence, K_2 can be closed into a Giroux's surface. $\square$

Theorem 7.3. *Let ξ be a non co-orientable Γ_1-structure on M^3 whose normal bundle ν embeds into τM. Then ξ is homotopic to a twisted open book foliation modified by suspension.*

Sketch of proof. Let $K = \Sigma \smallsetminus int\ D^2$ be the surface with a hole, where Σ was built in the first step 7.1; it meets the required assumptions for building a twisted open book.

The twisted open book built in the second step gives rise to a foliation $\mathcal{O}$. Indeed, as the binding B is co-orientable, it is allowed to spiral the pages around a tubular neighborhood of B. The tube itself is foliated by thick Reeb components. As in the co-orientable case, we can modify the open book foliation in a neighborhood of K using the representation φ, yielding the foliation $\mathcal{O}_\varphi$ and its associated regular Γ_1-structure ξ_φ. We have to prove that ξ and ξ_φ are homotopic. We may suppose that ξ is in Tsuboi form (7.1).

We observe that the total space of ν has a foliation $\mathcal{F}_0$ (unique up to isomorphism) transverse to the fibers, having the zero section as a leaf and whose all non trivial holonomy elements have order 2. It defines the trivial Γ_1-structure ξ_0 in the twisted sense. Using notation of 7.2, one can prove that $\xi|H'$ and $\xi_\varphi|H'$ are both homotopic to $\xi_0|H'$. Moreover, both homotopies coincide on the boundary $\partial H'$ (on H', it is sufficient to think of the case when φ is the trivial representation. Thus $\xi|H'$ and $\xi_\varphi|H'$ are homotopic relative to the boundary. $\square$

7.4. *Plane field homotopy class. By turbulizing $\mathcal{O}_\varphi$, it it possible to have the normal field in any homotopy class of embeddings $\nu \to \tau M$.*

Indeed, a curve in M is homotopic to a curve transverse to $\mathcal{O}_\varphi$ if and only if it does not twist ν. But these homology classes are exactly those which appear as a first difference homology class when comparing two embeddings $j_1, j_2 : \nu \to \tau M$, since a closed curve which twits ν is not a cycle in $H_1(M, \mathbb{Z}_{or(\nu)}) \cong H^2(M, \mathbb{Z}_{or(\nu * \otimes \tau M)})$.

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